Eur. Phys. J. D **22**, 423–431 (2003)
DOI: 10.1140/epjd/e2003-00028-6

The European Physical Journal D

Photon entanglement in the phase-space Q representation of quantum optics

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Received 10 August 2002 / Received in final form 7 November 2002
Published online 4 February 2003 –

Abstract. Several examples of photon entanglement are studied in the Q representation of quantum optics. In particular, the entangled states produced in parametric downconversion are studied in detail, and we determine the conditions for the violation of Bell's inequality. Our approach shows that photon entanglement is related to the existence of correlations between the quantum fluctuations of the electromagnetic field associated to different modes.

PACS. 03.65.Ud Entanglement and quantum nonlocality (e.g. EPR paradox, Bell's inequalities, GHZ states, etc.) – 42.50.Dv Nonclassical field states; squeezed, antibunched, and sub-Poissonian states; operational definitions of the phase of the field; phase measurements – 42.50.Lc Quantum fluctuations, quantum noise, and quantum jumps

1 Introduction

Entanglement is the characteristic trait of quantum mechanics [1], therefore it is worth studying it from a fundamental point of view. Also, entanglement is important for applications because it is a most relevant property in quantum information theory [2]. In the Hilbert-space formulation of quantum theory an entangled pure state of two systems is a linear combination of product vectors, but one which is not itself of product form and the generalization to arbitrary (*e.g.* mixed) states is possible. However it is difficult to determine whether a given state is entangled or not, and impossible to provide a unique measure of entanglement. In view of these facts it seems interesting to study entanglement in other formulations of quantum theory, which may provide a unified treatment for pure and mixed states and new physical insight in that important quantum phenomenon. This is the motivation for making a study of entanglement in a phase-space formulation. Some attention to this approach has already been given [3,4].

I shall restrict the study to electromagnetic radiation ("photon entanglement" in standard quantum language). This is because I think that understanding the physics of entanglement is easier in quantum field theory than in quantum mechanics, and electrodynamics is the best known example of field theory. Also the phase-space representations give a different perspective because they emphasize the wave aspects of radiation whilst the Hilbert-space formalism suggest a dual – waves and particles – character of light.

In Section 2 we recall the essentials of the Hilbert-space treatment of quantum optics, in order to stress a few points relevant for the rest of the article, and compare it with the phase-space representations, specially the Q representation. In Section 3 we consider the entanglement viewed in the Q representation, starting with the simple example of entanglement between the vacuum and a single-photon state. In Section 4 we discuss the entanglement of Gaussian states, where the contribution of the radiation both below and above the level of vacuum fluctuations may be most easily seen. Finally in Section 5 we summarize the results and discuss the difficulties of a purely wave theory of light suggested by the phase-space Q representation.

2 Hilbert-space and phase-space formalisms

2.1 Hilbert-space formulation of quantum optics

We shall work in the Heisenberg picture where the quantum formalism is most similar to the classical one. The electric field, $\mathbf{E}$, and the magnetic field, $\mathbf{B}$, of the radiation are time-dependent operators. The fields $\mathbf{E}$ and $\mathbf{B}$ may be expanded in normal modes, the coefficients of the expansion being the creation, $\hat{a}^{\dagger}_{\mathbf{k},\lambda}$, and annihilation, $\hat{a}_{\mathbf{k},\lambda}$, operators of photons. For instance in free space the (plane

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waves) expansion is

$$\mathbf{E}(\mathbf{r},t)=\sum_{\mathbf{k},\lambda}\sqrt{\frac{2\pi\omega}{L^3}}\,[\mathrm{i}\hat{a}_{\mathbf{k},\lambda}\mathbf{e}_{\mathbf{k},\lambda}\exp\left(\mathrm{i}\mathbf{k}\cdot\mathbf{r}-\mathrm{i}\omega t\right) + \mathrm{h.c.}],\quad \omega\equiv c|\mathbf{k}|, \tag{1}$$

$$\mathbf{B}(\mathbf{r},t)=\sum_{\mathbf{k},\lambda}\sqrt{\frac{2\pi\omega}{L^3}}\,\left[\mathrm{i}\omega^{-1}c\hat{a}_{\mathbf{k},\lambda}\mathbf{k}\times\mathbf{e}_{\mathbf{k},\lambda}\exp\left(\mathrm{i}\mathbf{k}\cdot\mathbf{r}-\mathrm{i}\omega t\right)\right. + \mathrm{h.c.}],\quad \omega\equiv c|\mathbf{k}|, \tag{2}$$

where $\mathbf{e}_{\mathbf{k},\lambda}$ is the polarization vector, L^3 the normalization volume and h.c. means hermitian conjugate. The creation and annihilation operators of photons fulfill the commutation relations

$$[\hat{a}_{\mathbf{k},\lambda},\hat{a}_{\mathbf{k}',\lambda'}]=\left[\hat{a}^{\dagger}_{\mathbf{k},\lambda},\hat{a}^{\dagger}_{\mathbf{k}',\lambda'}\right]=0,$$
$$\left[\hat{a}_{\mathbf{k},\lambda},\hat{a}^{\dagger}_{\mathbf{k}',\lambda'}\right]=\hbar\delta_{\mathbf{k}\mathbf{k}'}\delta_{\lambda\lambda'}, \tag{3}$$

where $\hbar$ is Planck's constant.

We have not written equations (1–3) in the conventional form, which is to include a factor $\sqrt{\hbar}$ in the plane waves expansions of the fields and define the creation and annihilation operators as dimensionless, so that $\hbar$ does not appear in the commutation relations (3). Our writing emphasizes the fact that the specifically quantum equations, which therefore should contain Planck's constant $\hbar$, are the commutation relations (3) and not the plane waves expansion. (But in the rest of the paper we shall put $\hbar=1$, except otherwise stated, so that our equations become the standard ones.)

The states of light may be written in terms of the Fock states, which form the basis for the quantum optical description of the light field. They are generated by applying the creation operators $\hat{a}^{\dagger}_{\mathbf{k},\lambda}$ to the Hilbert-space vector

$$|0\rangle=\prod_{\mathbf{k},\lambda}|0_{\mathbf{k},\lambda}\rangle, \tag{4}$$

which represents the vacuum. The whole Fock space is then spanned by the set of vectors

$$|\{n_{\mathbf{k},\lambda}\}\rangle=\prod_{\mathbf{k},\lambda}|n_{\mathbf{k},\lambda}\rangle=\prod_{\mathbf{k},\lambda}\frac{1}{\sqrt{n_{\mathbf{k},\lambda}!}}(\hat{a}^{\dagger}_{\mathbf{k},\lambda})^{n_{\mathbf{k},\lambda}}|0_{\mathbf{k},\lambda}\rangle, \tag{5}$$

which represents a state having $n_{\mathbf{k},\lambda}$ photons of wave number $\mathbf{k}$ and polarization λ. The latter index takes either the value 1 or 2. The most general *pure state* is a superposition of these, that is

$$|\Phi\rangle=\sum\phi(\{n_{\mathbf{k},\lambda}\})|\{n_{\mathbf{k},\lambda}\}\rangle,\quad \sum|\phi|^2=1. \tag{6}$$

The full set of quantum states is obtained by extending the set $|\Phi\rangle$ to *mixtures* of the form

$$\widehat{\rho}=\sum|\Phi\rangle W_{\Phi}\langle\Phi|,\quad 0\le W_{\Phi}\le 1,\quad \sum W_{\Phi}=1. \tag{7}$$

2.2 Phase-space P and Q representations

In the phase-space representations the creation and annihilation operators become complex variables corresponding to the amplitudes, $\{\alpha_{\mathbf{k},\lambda}\}$, of the modes, which are related to the fields by the expansions (1, 2). The states of light are phase-space pseudo-probability distribution. (Some phase-space functions may take negative values, in which case they cannot be probability distributions and this is why we call them pseudo-probabilities in general.) The best known phase-space function is the Wigner one, but in this paper we shall use only the P and the Q functions (see *e.g.* the book by Mandel and Wolf [5]). The former is related to the density operator in the form

$$\widehat{\rho}=\int P_{\rho}\left(\{\alpha_{\mathbf{k},\lambda}\}\right)\prod_{\mathbf{k},\lambda}|\alpha_{\mathbf{k},\lambda}\rangle\langle\alpha_{\mathbf{k},\lambda}|\mathrm{d}^2\alpha_{\mathbf{k},\lambda}. \tag{8}$$

where the integration should be performed with respect to the real and imaginary parts of every complex variable and

$$|\alpha\rangle=\exp\left(\alpha\hat{a}^{\dagger}-\alpha^{*}\hat{a}\right)|0\rangle =\sum_{n}\frac{\alpha^n}{\sqrt{n!}}\exp\left(-|\alpha^2|/2\right)|n\rangle, \tag{9}$$

is the coherent state for a mode. The Q function is defined as

$$Q_{\rho}(\{\alpha_{\mathbf{k},\lambda}\})=\langle\{\alpha_{\mathbf{k},\lambda}\}|\widehat{\rho}|\{\alpha_{\mathbf{k},\lambda}\}\rangle,$$
$$|\{\alpha_{\mathbf{k},\lambda}\}\rangle\equiv\prod_{\mathbf{k},\lambda}\frac{1}{\sqrt{\pi}}|\alpha_{\mathbf{k},\lambda}\rangle\;\cdot \tag{10}$$

The phase-space functions of the vacuum state are (I include here Planck's constant, see after Eqs. (3))

$$P_0(\{\alpha_{\mathbf{k},\lambda}\})=\prod_{\mathbf{k},\lambda}\delta^2(\alpha_{\mathbf{k},\lambda}),$$
$$Q_0(\{\alpha_{\mathbf{k},\lambda}\})=\prod_{\mathbf{k},\lambda}(\pi\hbar)^{-1}\exp\left(-\hbar^{-1}|\alpha_{\mathbf{k},\lambda}|^2\right). \tag{11}$$

The P function of the vacuum state, a product of Dirac's deltas, corresponds to the classical idea that the vacuum is empty. In contrast the Q function suggests that the vacuum corresponds to a random radiation (the zero-point field, ZPF). In general the P function, defined by (8), is not positive definite and therefore it cannot be interpreted as a probability distribution (frequently it is a generalized function more singular than Dirac's delta.) However there are some cases where it is nonnegative and the corresponding states are called *classical states of light* (see next section). Our writing of equation (11) shows that, in the classical limit $\hbar\to 0$, we have $Q_0\to P_0$ and both give a vacuum state which is empty. Thus in the Q representation *one major difference between classical and quantum theory of light is the existence, in the latter, of a ZPF in the vacuum state.*

For simplicity we shall consider in the following the ideal (*i.e.* unphysical) situation where only a single mode of the field is occupied (it "has photons"), so that the set $\{n_{\mathbf{k},\lambda}\}$ contains a single member, and the number states are designated simply $|0\rangle, |1\rangle, |2\rangle$... The full P or Q function will be given by the product of the single-mode function with $P_0(\alpha_{\mathbf{k},\lambda})$ or $Q_0(\alpha_{\mathbf{k},\lambda})$ for each of the "unoccupied" modes. The generalization to many modes should not be difficult.

From the definition (10) it is trivial to prove that the Q function is always positive so that it might be interpreted as a probability density (see the discussion section). With that interpretation quantum optics would become a pure wave theory of light where the quantum states are probability distributions for the realizations of the electromagnetic field. Furthermore the evolution of the free fields, $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$, (*i.e.* fields not interacting with charges) is given precisely by the Maxwell equations, as may be easily checked from (1) and (2) (with $\alpha_{\mathbf{k},\lambda}$ substituted for $\hat{a}_{\mathbf{k},\lambda}$.) However, the interaction with charges departs from what would be expected for a classical electromagnetic field. In particular the detection probability in a photon counter is not proportional to the value of the amplitude squared, but is given by (we put $\hbar = 1$ from now on)

$$p \propto \mathrm{Tr}\left[\hat{\rho}\hat{a}^{\dagger}\hat{a}\right] = \mathrm{Tr}\left[\hat{\rho}\left\{\hat{a}\hat{a}^{\dagger} - 1\right\}\right] = \int Q(\alpha)\left[|\alpha|^2 - 1\right]\mathrm{d}^2\alpha \equiv \langle|\alpha|^2 - 1\rangle_Q. \quad (12)$$

That is, a detector effectively "subtracts" the ZPF and detects only radiation which is above the ZPF background. This contrasts with the detection probability in the P representation, which does not require any subtraction. It is given by

$$p \propto \int P(\alpha)|\alpha|^2\mathrm{d}^2\alpha \equiv \langle|\alpha|^2\rangle_P. \quad (13)$$

In the Q representation there is no trace of corpuscules of light (photons). All we have are fluctuating electromagnetic fields. In particular the state with "n photons" in a single mode is represented by

$$Q_n(\alpha) = \frac{|\alpha|^{2n}}{n!}\exp\left(-|\alpha^2|\right), \quad (14)$$

which suggests that n-photon states are just a convenient basis of functions for the description of the probability distributions Q. In spite of the absence of "photons" the Q-representation is equivalent to the standard (Hilbert-space) one. In summary, the Hilbert-space representation of quantum optics suggests a dualistic (wave-particle) picture of light, whilst the Q representation suggests a purely wave picture, but both representations are physically equivalent (they predict the same results for all experiments).

2.3 Classical states of light

From equations (10, 8) it is straightforward to derive the following relation between the P and the Q representations (for a single mode)

$$Q(\alpha) = \int P(\beta)Q_0(\alpha - \beta)\mathrm{d}^2\beta. \quad (15)$$

The proof involves inserting (8) into (10) and then using the scalar product of two coherent state-vectors (in the same mode), which is [5]

$$|\langle\alpha|\beta\rangle|^2 = \exp\left(-|\alpha - \beta|^2\right).$$

Taking into account the form of the Q-function of the vacuum, equation (11), we get finally equation (15), which means that the Q-function of a state is the convolution of the P-function of that state with the Q-function of the vacuum state. For completeness we state, without proof, similar relations involving the Wigner function, $W(\alpha)$,

$$W(\alpha) = \frac{1}{\pi}\int P(\beta)\exp\left(-2|\alpha - \beta|^2\right)\mathrm{d}^2\beta, \quad (16)$$
$$Q(\alpha) = \frac{1}{\pi}\int W(\beta)\exp\left(-2|\alpha - \beta|^2\right)\mathrm{d}^2\beta.$$

The relation (15) provides an interesting picture if we accept the Q-function as a probability distribution, namely the total field amplitude is the sum of two amplitudes which are independent random variables. (As is well-known the probability distribution of a random variable which is the sum of two independent ones is the convolution of their distributions.) In this form the classical states of light appear as those where a signal radiation is superimposed incoherently to the ZPF. When this is the case, all optical phenomena are associated with the signal alone, and the ZPF may be ignored altogether, as is the situation in classical optics. In particular detectors remove precisely the ZPF, see (12).

The above interpretation of the classical states of light may be illustrated with the coherent state (9), which is the only pure quantum state which is *classical*. Its P and Q functions are (here we put $\hbar = 1$)

$$P_a(\alpha) = \delta^2(\alpha - a),$$
$$Q_a(\alpha) = (1/\pi)\exp\left(-|\alpha - a|^2\right). \quad (17)$$

The Q function plus equation (15) suggest that this is a state where a deterministic signal (with probability distribution given by $P_a(\alpha)$) is superimposed incoherently to the ZPF (with probability distribution given by the Q function (11).

3 Entanglement in the Q representation

3.1 An example of entangled pure state: A single photon with the vacuum

For the purposes of this subsection a convenient definition of entanglement, although not the most general one (see (28) below), is the following. Two systems, ϕ and χ, localized in two distant regions, say near the points $\mathbf{r}_1$ and $\mathbf{r}_2$, are entangled if the state vector of the composite system is

$$|\psi\rangle = c_1|\phi, \mathbf{r}_1\rangle \otimes |\chi, \mathbf{r}_2\rangle + c_2|\chi, \mathbf{r}_1\rangle \otimes |\phi, \mathbf{r}_2\rangle, \tag{18}$$

with a standard notation. Here c_1 and c_2 are complex numbers, both different from zero, such that $|c_1|^2 + |c_2|^2 = 1$. As is well-known the entangled state (18) is quite different from the mixed state represented by the density operator

$$\begin{aligned}\widehat{\rho} = {} & |c_1|^2 \, |\phi, \mathbf{r}_1\rangle\langle\phi, \mathbf{r}_1| \otimes |\chi, \mathbf{r}_2\rangle\langle\chi, \mathbf{r}_2| \\ & + |c_2|^2 \, |\chi, \mathbf{r}_1\rangle\langle\chi, \mathbf{r}_1| \otimes |\phi, \mathbf{r}_2\rangle\langle\phi, \mathbf{r}_2|. \end{aligned} \tag{19}$$

In the case of radiation, localization is achieved by constructing wavepackets consisting of a sum (integral) of plane waves, with wavevectors centered around a given one, say $\mathbf{k}$. Consequently any localized state should contain many modes of the radiation. But we may simplify the argument by using a single mode for each system, that is the mode corresponding to wavevector $\mathbf{k}$ and polarization λ. Then we shall speak of entanglement in momentum (instead of position) and write, in place of (18) and (19), the following

$$|\psi\rangle = c_1|\phi, 1\rangle \otimes |\chi, 2\rangle + c_2|\chi, 1\rangle \otimes |\phi, 2\rangle, \tag{20}$$

$$\begin{aligned}\widehat{\rho} = {} & |c_1|^2 \, |\phi, 1\rangle\langle\phi, 1| \otimes |\chi, 2\rangle\langle\chi, 2| \\ & + |c_2|^2 \, |\chi, 1\rangle\langle\chi, 1| \otimes |\phi, 2\rangle\langle\phi, 2|, \end{aligned} \tag{21}$$

where 1 (2) stands for $\{\mathbf{k}_1, \lambda = 1\}$ ($\{\mathbf{k}_2, \lambda = 1\}$). From (20) and (21) it is straightforward to get the Q functions using equation (10), but for the sake of clarity we shall firstly work a particular example.

Our example is the simplest possible entanglement, namely one between the vacuum and a single-photon system. It may be experimentally realized by sending a single-photon signal to a beam-splitter, an experiment which has been performed in order to show the wave-particle properties of light [6]. The state in two of the outgoing modes of the beam splitter may be represented by equation (20), ϕ being the vacuum state and χ a single-photon state. It may be shown that, if two detectors are placed in the appropriate outgoing channels, the photon is detected in only one of them, which is interpreted as a "particle" behaviour of light. This would also happen if the state were a mixture like (21). However if the beams are recombined the entangled state exhibits interference, a wave property which clearly distinguishes it from the mixture [6].

The passage to the Q representation is straightforward taking into account the representatives of the vacuum, (11), and the single-photon state, (14) with $n = 1$. It is easy to see that the radiation incoming to the beam splitter is represented by

$$Q(\{\alpha_{\mathbf{k},\lambda}\}) = Q_0(\{\alpha_{\mathbf{k},\lambda}\}) \, |\beta|^2 \,, \tag{22}$$

β being the amplitude of the radiation mode containing the photon. In order to calculate the state of the outgoing radiation we should use the rules of classical wave optics, that is the radiation arriving at the beam-splitter in every mode is partially transmitted and partially reflected. Also, if an outgoing channel contains radiation coming from two different modes, we shall add their amplitudes (not their intensities!). The reason for using the rules of classical optics is that, in the absence of absorption or emission, radiation fields propagate according to Maxwell equations, in both the Hilbert-space representation (see (1) and (2)) and the Q representation (see Sect. 2). Thus, as we have radiation in every incoming radiation mode (a single-photon field in one mode and the ZPF in all other modes), we shall have radiation from two incoming channels in every outgoing channel of the beam splitter. In particular the amplitude β of the mode containing the photon is partially transmitted and partially reflected, giving in the appropriate outgoing channels the amplitudes $\beta/\sqrt{2}$ and $\beta\mathrm{i}/\sqrt{2}$, respectively (assuming a 50–50 beam-splitter). But these amplitudes should be added to the ones reflected and transmitted, respectively, from the ZPF in another incoming channel, say with amplitude γ. As a result there are two outgoing modes with amplitudes

$$\begin{aligned}\mu &= \frac{1}{\sqrt{2}} \left(\beta + \mathrm{i}\gamma\right), \\ \nu &= \frac{1}{\sqrt{2}} \left(\gamma + \mathrm{i}\beta\right). \end{aligned} \tag{23}$$

In all other outgoing channels there are amplitudes like (23) with both β and γ corresponding to incoming ZPF radiation. It may be shown that the radiation in every one of these outgoing channels is again pure ZPF.

It may be realized that a similar addition rule, of the creation or annihilation operators, is necessary when we work in the Hilbert-space representation [7]. In that representation the introduction of the operators of the "unactivated" (*i.e.* without photons) modes is required in order to preserve the commutation relations for the outgoing modes, which corresponds to preserving the randomness (due to the ZPF) in the Q representation.

Now the Q-function of the outgoing radiation (the entangled state of vacuum and single-photon signal) is simply equation (22) written in terms of μ and ν, instead of β. That is

$$Q_{\mathrm{ent}}(\{\alpha_{\mathbf{k},\lambda}\}) = \frac{1}{2} \, |\mu - \mathrm{i}\nu|^2 \, Q_0(\{\alpha_{\mathbf{k},\lambda}\}). \tag{24}$$

In contrast, a mixture would be represented by

$$Q_{\mathrm{mix}}(\{\alpha_{\mathbf{k},\lambda}\}) = \frac{1}{2} \left(|\mu|^2 + |\nu|^2\right) Q_0(\{\alpha_{\mathbf{k},\lambda}\}), \tag{25}$$

as may be easily derived from (21).

We see that, if we interpret the Q function as a probability distribution, both the entangled state and the mixed state represent situations where there is a correlation between the field amplitudes of two radiation modes. The main difference is that the mixture just involves the correlation of the intensities whilst the entangled state contains a more subtle correlation involving also the phases. Therefore we are tempted to say that, in the Q-representation, the essential difference between entangled state and mixture is that the former involves correlation of the phases. This fits with the well-known fact that an average over the relative phase of c_1 and c_2 in (18) produces (19). However, looking more closely at the situation shows some problems for this interpretation, as is discussed in the following.

The anticorrelation after the beam splitter (*i.e.* the fact that the photon is detected only in one outgoing channel) is explained in the Q representation as follows. We shall calculate the probability of a coincidence count in both channels using a straightforward generalization of (12). We have

$$p_{12} \propto \int \mathrm{d}^2\mu \int \mathrm{d}^2\nu\, Q(\mu,\nu) \left(|\mu|^2 - 1\right) \left(|\nu|^2 - 1\right), \quad (26)$$

where we have ignored all modes except the two relevant ones (*i.e.* the function $Q(\mu,\nu)$ is the result of integrating the full Q-function over the irrelevant modes, which contain just ZPF). We see that for both functions (24) and (25) the result of the integration (26) is zero. In the case of the mixed state the interpretation is clear: the detector subtract the ZPF and there is only one mode with radiation above the ZPF (we might say that the photon is in one beam, but we do not know which). However, in the case of the entangled state the explanation is more subtle and follows from equation (23). We see that the outgoing amplitudes μ and ν result from an interference between the incoming ones β (of the single-photon beam) and γ (from the ZPF). But conservation of energy requires that, if μ is large (constructive interference), necessarily ν is small (destructive interference). Consequently only one of the amplitudes, and not both, can be above "the level of the ZPF" which is required to produce a detection event. In summary, in the interpretation of light as pure waves, with a Q probability distribution, the anticorrelation after a beam splitter is just a consequence of the existence of a ZPF and not a "particle" property of light.

The generalization of equations (22–26) is straightforward. We consider a general entangled pure state represented by equation (18), whose Q function is

$$Q_{\mathrm{ent}}(\alpha,\beta) = \left|c_1\phi(\alpha)\chi(\beta) + c_2\phi(\beta)\chi(\alpha)\right|^2 Q_0, \quad (27)$$

where α or β label collectively all the modes used in order to localize a wavepacket around point $\mathbf{r}_1$ or $\mathbf{r}_2$, respectively (we assume that the two sets of modes are disjoint). The functions $\phi(\alpha)$, $\chi(\beta)$, $\phi(\beta)$ and $\chi(\alpha)$ may be easily related to the Q-representatives of the states $|\phi,\mathbf{r}_1\rangle$, $|\chi,\mathbf{r}_2\rangle$, $|\chi,\mathbf{r}_1\rangle$ and $|\phi,\mathbf{r}_2\rangle$, respectively. In contrast the Q-function of the mixed state will be

$$Q_{\mathrm{mix}}(\alpha,\beta) = \Big(|c_1|^2 |\phi(\alpha)|^2 |\chi(\beta)|^2 + |c_2|^2 |\chi(\alpha)|^2 |\phi(\beta)|^2\Big) Q_0.$$

After that we see that the essential difference between mixture and entangled state is the fact that the ZPF is correlated in a nontrivial manner (involving the phases). It is not just that there is a phase correlation because also in classical optics the phases may be correlated. For instance if we send a coherent monochromatic beam (*e.g.* from a laser) with complex amplitude a into a beam splitter, the probability distribution of the amplitudes in the incoming and outgoing beams are

$$P_{\mathrm{in}} = \delta^2(\alpha - a),$$
$$P_{\mathrm{out1}} = \delta^2\left(\mu - \frac{a}{\sqrt{2}}\right),$$
$$P_{\mathrm{out2}} = \delta^2\left(\nu + \frac{\mathrm{i}a}{\sqrt{2}}\right)$$

which clearly involves a correlation of the phases like in the entangled state (24). Indeed, the two outgoing beams are able to interfere. We conclude that phase correlation is a wave property which may be present in both classical and quantum optics. Therefore the characteristic trait of entanglement cannot be phase correlation alone, but the fact that the phase correlation involves the ZPF. This is clarified using the particular case of Gaussian states of light as is made in Section 4.

3.2 Mixed states. Separability and classicality

Two systems, 1 and 2, in a state described by a density operator ρ, are said entangled if ρ is not separable. The state ρ is separable if, and only if, it can be expressed in the following form

$$\rho = \sum_j w_j \rho_{j1} \otimes \rho_{j2}, \quad (28)$$

where we assume that ρ_{j1} and ρ_{j2} are normalized states of the systems 1 and 2, respectively, and $w_j \geq 0$ satisfy $\sum_j w_j = 1$.

A full characterization of separability, as defined in (28), using phase-space representations will not be given here. We shall rather investigate the relation between classicality and separability. It is well-known [3] and trivial to prove that, if the state ρ is classical, it is separable and both local states, ρ_1 and ρ_2, are also classical. ("P-representable" is often used instead of "classical", but we prefer here the latter name in order to emphasize the physical, rather than the mathematical, aspects). We call local state of system 1 (2) the one associated to the density operator obtained by taking the partial trace with respect to 2 (1). That is, the local states ρ_1 and ρ_2 will be

$$\rho_1 = \mathrm{Tr}_2\rho,\ \rho_2 = \mathrm{Tr}_1\rho.$$

If the state ρ is not classical it is not necessarily entangled. For instance the state (20) with $c_2 = 0$ would be separable, but not classical. Therefore classicality is stronger than separability. However one is tempted to conjecture that for some families of states, like Gaussian ones [4], both conditions are equivalent. In the following section we shall show that the conjecture for Gaussian states is not true. In any case classicality and separability are closely related.

4 Gaussian states of light

4.1 States with two modes

Gaussian states of light are defined as those whose characteristic functions are Gaussian, an equivalent definition being that the Q function is Gaussian. General criteria of separability for two-modes Gaussian states have been recently given [3,8] and a generalization to many modes also exists [9]. Here we shall study a restricted family of two-mode states whose single-mode ones are classical.

We shall illustrate the relation between classicality and separability using a Gaussian two-modes state having the following Q function

$$Q_{12}(\alpha,\beta)=\frac{(1-x^2)K^2}{\pi^2}\exp\left\{-K\left[|\alpha|^2+|\beta|^2\right.\right.$$
$$\left.\left.+\,x\left(\alpha\beta+\alpha^*\beta^*\right)\right]\right\},$$
$$K\equiv\frac{1}{(n+1)\left(1-x^2\right)},\ n\geq 0,\ |x|<1, \tag{29}$$

where n is the mean number of photons per mode and x a correlation parameter. The associated local single-mode states have the following P_1 and Q_1 functions

$$P_1(\alpha)=\frac{1}{\pi n}\exp\left\{-\frac{1}{n}|\alpha|^2\right\},$$
$$Q_1(\alpha)=\frac{1}{\pi\,(n+1)}\exp\left\{-\frac{1}{(n+1)}|\alpha|^2\right\}, \tag{30}$$

and similar for P_2 and Q_2. (If $n\to 0$ P_1 and P_2 become Dirac's deltas, see (11).)

For the study of Gaussian states it is convenient to introduce the variance matrix, V, of the variables $\xi_1=(1/\sqrt{2})\mathrm{Re}\,\alpha$, $\xi_2=(1/\sqrt{2})\mathrm{Im}\,\alpha$, $\xi_3=(1/\sqrt{2})\mathrm{Re}\,\beta$, $\xi_4=(1/\sqrt{2})\mathrm{Im}\,\beta$. It is easy to see that the only nonzero elements are

$$V_{jj}=\left\langle\xi_j^2-\frac{1}{2}\right\rangle_Q=n+\frac{1}{2},$$
$$V_{j,j\pm 2}=\left\langle\xi_j\xi_{j\pm 2}\right\rangle_Q=(-1)^j\,x\,(n+1)\,,$$

where

$$\left\langle F\left(\{\xi_j\}\right)\right\rangle_Q\equiv\int FQ\left(\alpha,\beta\right)\mathrm{d}^2\alpha\mathrm{d}^2\beta.$$

In the Hilbert-space formalisms the variances are usually defined taking the creation and annihilation operators in symmetrical ordering, which is equivalent to averaging with the Wigner function. Now averaging ξ_j^2 with the Wigner function corresponds to averaging $\xi_j^2-1/2$ with the Q function, associated to antinormal ordering. Writing the variance matrix V in block form we obtain from (29)

$$V=\begin{pmatrix}A & C\\ C^T & B\end{pmatrix},$$
$$A=B=\left(n+\frac{1}{2}\right)I,\quad C=-x\,(n+1)\,I, \tag{31}$$

where I is the identity matrix.

Equation (29) is not a valid Q function for all values of x and the possible values may be got, for instance, using Simon's criterion [3], derived from the uncertainty principle,

$$V+\frac{\mathrm{i}}{2}\Omega\geq 0,$$
$$\Omega\equiv\begin{pmatrix}J & 0\\ 0 & J\end{pmatrix},\quad J\equiv\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}.$$

We obtain the condition

$$|x|\leq\sqrt{\frac{n}{n+1}}\,. \tag{32}$$

Now the relevant result for our purposes is that *the state with Q function* (29) *is separable if, and only if, it is classical.*

In order to prove the assertion we firstly determine whether (29) corresponds to a classical state. For this purpose we construct a P function which, by convolution with the vacuum Q function (see (15)), may give (29). (An alternative procedure, maybe easier but less direct, would be to use the following condition on the variance matrix: $V-\frac{1}{2}I\geq 0$, which easily derives from (16).) The P function should have the form

$$P_{12}(\alpha,\beta)=\frac{r^2-s^2}{\pi^2}\exp\left\{-r\left(|\alpha|^2+|\beta|^2\right)\right.$$
$$\left.-\,s\left(\alpha\beta+\alpha^*\beta^*\right)\right\}, \tag{33}$$

r and s being two real numbers, fulfilling $0<|s|\leq r$, in order that P_{12} is normalizable. The case $r\to 0$ is also possible, leading to the distribution (35), see below. The convolution of (33) with the Q function of the two-modes vacuum state, $Q_0(\alpha)Q_0(\beta)$, gives (29) provided that

$$r=\frac{n}{n^2-x^2\,(n+1)^2},$$
$$s=\frac{(n+1)x}{n^2-x^2\,(n+1)^2}\,.$$

Hence the Q function (29) will represent a classical state if, and only if, $r\geq 0$, which leads to

$$\text{classical: } |x|\leq\frac{n}{n+1},$$
$$\text{nonclassical: } |x|>\frac{n}{n+1}\,. \tag{34}$$

The equality sign should be understood as a limit $|x|\to n/(n+1)$, with n fixed, taken from below. It corresponds to a classical state with P function

$$P_{12}(\alpha,\beta)=\frac{1}{\pi n}\exp\left(-\frac{1}{n}|\alpha|^2\right)\delta^2\left(\alpha\pm\beta^*\right), \tag{35}$$

the + (−) sign corresponding to positive (negative) x. Taking into account the condition (32) we see that the state is nonclassical in the range $n/(n+1)<|x|\leq\sqrt{n/(n+1)}$.

Now we study the separability of (29) using Simon's criterion [3]. It tells that the state is separable if, and only if, the following inequality holds true

$$\det A \det B + \left(\frac{1}{4} - |\det C|\right)^2 - \mathrm{Tr}(AJCJBJC^T J) \geq \frac{1}{4}\left(\det A + \det B\right).$$

Hence we get

$$\begin{aligned} &\text{separable: } |x| \leq \frac{n}{n+1}, \\ &\text{entangled: } \frac{n}{n+1} < |x| \leq \sqrt{\frac{n}{n+1}}, \end{aligned} \tag{36}$$

the latter inequality deriving from (32). A comparison with (34) proves our statement.

An interpretation of these results emerges in our approach. Remembering equation (15), we see that (29) might be considered the probability distribution of the sum of two random variables representing the signal and the ZPF, respectively. In the signal, whose distribution is (33), there is correlation between the two modes. In contrast in the ZPF, whose distribution is (11), all modes are uncorrelated (the distribution is a product of single-mode terms). In summary, *if the state is separable (entangled) we have a correlation involving only the signal (both the signal and the ZPF).*

The study may be extended to general two-modes Gaussian states. Modulo an appropriate local linear unitary Bogoliubov operation [8], the most general variance matrix may be written in block form (see (31)) with

$$A = aI, \quad B = bI, \quad C = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}.$$

Then, with the method used above, it is straightforward to show that the state is classical if, and only if,

$$\text{classical: } \quad c_j^2 \leq \left(a - \frac{1}{2}\right)\left(b - \frac{1}{2}\right), \; j = 1, 2. \tag{37}$$

From Simon's criterion it is separable if, and only if,

separable:

$$ab\left(c_1^2 + c_2^2\right) - c_1^2 c_2^2 + \frac{1}{2}|c_1 c_2| \leq \left(a^2 - \frac{1}{4}\right)\left(b^2 - \frac{1}{4}\right). \tag{38}$$

We see that separability is a sufficient condition for classicality if $c_1 = \pm c_2$, but not in general. For instance, if $c_2 = 0$, (38) gives the condition

separable:

$$c_1^2 \leq \left(a - \frac{1}{2}\right)\left(b - \frac{1}{2}\right)\left(1 + \frac{1}{2a}\right)\left(1 + \frac{1}{2b}\right), \; c_2 = 0,$$

to be compared with the more restrictive

$$\text{classical: } c_1^2 \leq \left(a - \frac{1}{2}\right)\left(b - \frac{1}{2}\right), \; c_2 = 0,$$

derived from (37).

4.2 States produced in parametric downconversion

The example of the previous subsection is interesting because it is closely related to the state of light which is produced in the process of parametric downconversion. That process has been the most widely used source of entangled photon pairs during the last two decades. We will study entanglement in momentum and polarization, a situation which is produced in type II parametric downconversion [10]. The physical process is a many-modes one, but we shall consider just four modes, which is a typical approximation in the analysis of the experiments. In the Hilbert-space formulation the two-photon state is usually represented by the state-vector

$$|\Phi_1\rangle = \frac{1}{\sqrt{2}}\left(\hat{a}_{\mathrm{H}}^{\dagger}\hat{b}_{\mathrm{V}}^{\dagger} + \hat{a}_{\mathrm{V}}^{\dagger}\hat{b}_{\mathrm{H}}^{\dagger}\right)|0\rangle,$$

where the subindex H (V) means horizontal (vertical) polarization and the letters a and b label two light beams (with different momenta). Actually the photons are produced by spontaneous emission so that, in any finite time interval, we may have either one photon pair, or two pairs, ... or none. Consequently the appropriate representation of the state is the density matrix

$$\hat{\rho} = \sum_n w_n |\Phi_n\rangle\langle\Phi_n|, w_n \geq 0, \qquad \sum_n w_n = 1. \tag{39}$$

As the state is Gaussian in parametric downconversion [11,12], we may rewrite the density matrix, introducing a real parameter x, in the form

$$\begin{aligned} \hat{\rho} = N \exp\left[x\left(\hat{a}_{\mathrm{H}}^{\dagger}\hat{b}_{\mathrm{V}}^{\dagger} + \hat{a}_{\mathrm{V}}^{\dagger}\hat{b}_{\mathrm{H}}^{\dagger}\right)\right]|0\rangle\langle 0| \\ \times \exp\left[x\left(\hat{a}_{\mathrm{H}}\hat{b}_{\mathrm{V}} + \hat{a}_{\mathrm{V}}\hat{b}_{\mathrm{H}}\right)\right], \end{aligned} \tag{40}$$

where N is a normalization constant.

It is trivial to get the Q function of the state (40) using the definition (10). We obtain

$$\begin{aligned} Q_{12}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \frac{\left(1 - x^2\right)^2}{\pi^4} \exp\Big[- |\boldsymbol{\alpha}|^2 - |\boldsymbol{\beta}|^2 \\ + x\left(\boldsymbol{\alpha}\cdot\boldsymbol{\beta} + \boldsymbol{\beta}^*\cdot\boldsymbol{\alpha}^*\right)\Big], \end{aligned} \tag{41}$$

where we define the vector amplitudes by

$$\boldsymbol{\alpha} \equiv \left(\alpha_{\mathrm{H}}, \alpha_{\mathrm{V}}\right), \; \boldsymbol{\beta} \equiv \left(\beta_{\mathrm{V}}, \beta_{\mathrm{H}}\right).$$

This function may be seen as the product of two similar to Q_{12} of (29), both with $A = 1$, the first involving α_{H} and β_{V}, and the second involving α_{V} and β_{H}. The expectation value of the number of photons in each beam, $2n$, may be calculated using (12) for every polarization. (In order make the comparison with the results of the previous subsection, here we shall label n the average number of photons with given polarization in *every* beam, that is $4n$ photons in the two beams together.) We get for the first beam

$$2n = \int Q_{12}(\boldsymbol{\alpha}, \boldsymbol{\beta}) \left(|\alpha_{\mathrm{H}}|^2 + |\alpha_{\mathrm{V}}|^2 - 2 \right) \times \mathrm{d}^2\alpha_{\mathrm{H}} \mathrm{d}^2\alpha_{\mathrm{V}} \mathrm{d}^2\beta_{\mathrm{H}} \mathrm{d}^2\beta_{\mathrm{V}} = \frac{2x^2}{1-x^2}, \tag{42}$$

and the same value for the other beam. Equation (42) implies

$$|x| = \sqrt{\frac{n}{n+1}},$$

which, looking at (32), shows that $|x| < 1$ attains the maximum possible value. Hence we get

$$\frac{n}{n+1} < \sqrt{\frac{n}{n+1}} = |x|, \tag{43}$$

which, by comparison with (36), proves that the state (40) is indeed entangled for any $n > 0$, which implies $x \neq 0$.

4.3 Bell's inequalities for Gaussian states

Now we shall study if the entangled states with density operator (40) violate a Bell inequality. As is well-known, the violation of a Bell inequality is a sufficient, but not necessary, condition for entanglement. In actual experiments genuine Bell inequalities, derived using only general properties of local hidden variables, have never been tested. All tested inequalities involve auxiliary assumptions, a fact qualified as "existence of loopholes". The violation of one of such inequalities does not imply the refutation of local realism [13]. Nevertheless we may consider the experiments as valid *tests of entanglement*. In practice any test involves measuring a coincidence detection rate as a function of some controllable angular parameter, ϕ. The inequalities are violated if the measured coincidence rate, R_{12}, is of the form

$$R_{12} = const. \times (1 + V \cos\phi), \tag{44}$$

with the visibility or contrast, V, greater than $\sqrt{2}/2 \simeq 0.71$.

We shall consider that the amplitudes $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ correspond to two polarized light beams arriving at two polarization analyzers at angles ϕ_1 and ϕ_2, respectively. In what follows we may ignore the ZPF in all modes except those included in (41). Consequently we assume that the amplitudes emerging from the polarizers are given by Malus law, that is

$$\lambda = (\boldsymbol{\alpha} \cdot \mathbf{u}_1), \quad \mu = (\boldsymbol{\beta} \cdot \mathbf{u}_2),$$

the vector $\mathbf{u}_1$ having components $(\cos\phi_1, \sin\phi_1)$ and similar for $\mathbf{u}_2$. Note that the scalar amplitudes λ and μ correspond to modes with polarization in the directions of $\mathbf{u}_1$ and $\mathbf{u}_2$, respectively. As said above we ignore the modes with polarization perpendicular to $\mathbf{u}_1$ or $\mathbf{u}_2$, which contain just ZPF.

The coincidence detection rate in two detectors placed after the polarizers may be calculated by a straightforward generalization of (12), namely

$$R_{12} \propto \left\langle \left(|\lambda|^2 - 1 \right) \left(|\mu|^2 - 1 \right) \right\rangle_Q \equiv \int Q_{12}(\boldsymbol{\alpha}, \boldsymbol{\beta}) \left(|\lambda|^2 - 1 \right) \left(|\mu|^2 - 1 \right) \times \mathrm{d}^2\alpha_x \mathrm{d}^2\alpha_y \mathrm{d}^2\beta_x \mathrm{d}^2\beta_y.$$

In order to perform the integral we use a well-known property of the correlations of Gaussian random variables, namely

$$\left\langle \left(|\lambda|^2 - 1 \right) \left(|\mu|^2 - 1 \right) \right\rangle = \left\langle |\lambda|^2 - 1 \right\rangle \left\langle |\mu|^2 - 1 \right\rangle + |\langle \lambda\mu^* \rangle|^2 + |\langle \lambda\mu \rangle|^2. \tag{45}$$

The first term on the right hand side is just the square of one half the number of photons involved, this given by (42). The second term is zero, and the third one may be calculated by integration, leading to

$$|\langle \lambda\mu \rangle| = \frac{x \cos(\phi_1 - \phi_2)}{1 - x^2}.$$

As a result we get

$$R_{12} \propto \frac{x^2 \left(1 - 2x^2\right)}{\left(1 - x^2\right)^2} \left[1 + V \cos(2\phi_1 - 2\phi_2)\right],$$

where the visibility is

$$V = \frac{1}{1 + 2x^2}.$$

We see that Bell's inequality may be violated (the visibility surpasses the limit 0.71) if $|x| < 0.45$, which corresponds to $n < 0.60$ photons in each beam.

The fact that the correlation parameter $|x|$ must be small in order to have a violation of the Bell inequality seems counterintuitive, but it is a consequence of the fact that high correlations are only possible for high photon numbers (see (43)), which correspond to more "classical" light. Thus we see that there is a trade-off between increasing the correlation and increasing the light intensity, and this is what makes so difficult to implement loophole-free tests of the Bell inequality using parametric downconverted light.

5 Discussion

The main result of the present paper has been to show that a close connection exists between two typically quantal phenomena: entanglement and vacuum fluctuations. I think that the approach used may provide a new perspective on both phenomena. Also the results obtained suggest that the use of phase-space representations in the study of photon entanglement may be profitable.

An interpretation of the Q function as a probability distribution, looking at light as a purely wave phenomenon, has been used sometimes in this paper. However it is necessary to point out there are important difficulties for that interpretation. Indeed from the definition (10) $Q(\alpha)$ is the probability density that the system is in the coherent state $|\alpha\rangle$. More properly, we may define a POVM (positive operator-valued measure) whose elements are, up to an overall constant of proportionality, equal to the coherent state projectors. However, as these projectors are not orthogonal, we would not say that it is really the case that it is a probability distribution for the field amplitudes. (A more appropriate phase-space function would be the Wigner function, were not for the difficulty that is not positive definite for all quantum states [14].) In consequence the wave interpretation of light mentioned in the present article may be seen as just an useful aid to the intuition in some instances.

I acknowledge financial support from DGICYT, Project No. PB-98-0191 (Spain). I thank the comments and suggestions of two anonymous referees, and Matilde Leal for a critical reading of the manuscript.

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